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Modelling Life Table Functions

Introduction

EARLIER attempts to develop mathematical models to describe the patterns of variation of life table functions may be traced back to Gompertz who was perhaps the first to propose his thesis that man's ability to resist death decreases with increase in age. This thesis translated in mathematical terms resulted in a function named after him which shows the pattern of mortality with advance in age in terms of a geometric progression. The model is not suited to the early childhood ages, since, in this age range, mortality declines rather than increases with age. Also, the model does not fit well at the upper extreme of the age range although, the fit is somewhat better for the generational than for the cross-sectional mortality. Makeham (1860) suggested that an extraneous or a chance factor, independent of age, also contributed to mortality and he added a constant term to the Gompertz' model.

Makeham's formula based on three parameters produces a closer fit than that of Gompertz, especially for young adult ages, but considerable discrepancies are still found at other ages. To overcome the difficulty, Perks (1932) modified the Gompertz formula by adding still another parameter and the problems still remained. Recently, Pollard (1979) has experimented with a six parameter model for the complete life span, but the number of parameters makes the model somewhat less than satisfactory. Earlier, Mitra (1974) had reported on the goodness of fit of the logarithm of life expectancy when represented by a quadratic function of age. It appears that there is still considerable scope for further research in the modelling of life table functions and the results of such an endeavor are presented in this paper.

The Model

In our attempt to develop mathematical expressions for the life table functions, we begin by noting that the survivorship function $l(x)$ is a monotonically declining function of age x with $l(0) = 1$ and $l(\alpha) = 0$, where α is the length of the life span. Further, we also recognize the well-known behaviour of the force of mortality $\mu(x) = - dln(x)/dx$ in the age range which in general terms can be described as follows. The function $\mu(x)$ begins its decline from a very large value at $x = 0$ to attain a minimum somewhere in the teens (Mitra, 1977); it increases thereafter and becomes very large as it approaches the end of the life span. Thus, any mathematical expression representing a life table function must, at the very least, exhibit the essential characteristics mentioned above.

Since, by definition,

$$lnl(x) = -\int_0^x \mu(x)dx \tag{1}$$

it seems logical that we search for a function $f(x)$ such that

$$l(x) = e^{-(fx)} \tag{2}$$

where

$$\int_0^x \mu(x)dx = f(x). \tag{3}$$

This function $f(x)$ must meet the following condition, namely,

(*a*) $f(0) = 0$, so that $l(0) = 1$,

(*b*) $f(\alpha) = \infty$, so that $l(\alpha) = 0$, and

(*c*) $f'(x)$ or $\mu(x)$ is uniformly positive and assumes a minimum value at an appropriate age and very large values at the two extremes.

Clearly,

$$f(x) = \frac{Ax^m}{(\alpha - x)^n} \qquad A, m, n > 0 \tag{4}$$

can be immediately seen to meet the first and the second condition. As regards the third, we first differentiate (4) to get

$$f'(x) = \mu(x) = \frac{mAx^{m-1}}{(\alpha - x)^n} + \frac{nAx^m}{(\alpha - x))^{n+1}} \tag{5}$$

The force of mortality function generated from our model and shown in (5) consists of two components. It may be noted that the second component, namely, $nAx^m l(a-x)^{n+1}$ uniformly increases with age starting from the lowest value of 0 at age 0. Interestingly enough, this component seems to match the pattern postulated by Gompertz on the assumption that the ability to withstand death increases with age. However, this is not enough, since, as is well known, mortality continues to decrease from a relatively high value at age 0 until a certain age, somewhere in the teens, is reached. This pattern of the variation is taken care of by the first term of the model equation, namely, $mAx^{m-1}/(\alpha-x)^n$. For $m < 1$, it may be seen that this component continues to decline with increase in age from 0 to $\alpha(1-m)/(n-m+1)$. It begins to increase thereafter and in that way gives an extra boost to the already rising rates generated by the second component.

Thus we see from (5) that $f'(x) = \mu(x)$ is uniformly positive and for $n > 0$, becomes larger and larger as x approaches α. In fact, $\mu(\alpha) = \infty$. For the same to happen at $x = 0$, and because $l(0) = 1$, we must have

$$0 < m < 1. \tag{6}$$

In that case, $f'(x)$ must have at least one minimum, to determine which we first rewrite (5) as

$$f'(x) = \mu(x) = \left(\frac{m}{x} + \frac{n}{\alpha-x}\right) f(x) \tag{7}$$

for $0 < x < \alpha$. Differentiating (7) with respect to x we get

$$f''(x) = \mu'(x) = \left[\left(\frac{m}{x} + \frac{n}{\alpha-x}\right)^2 - \frac{m}{x^2} + \frac{n}{(\alpha-x)^2}\right] f(x). \tag{8}$$

The values of x corresponding to the minimum value of $\mu(x) = f'(x)$ may next be obtained by equating (8) to zero and solving the resulting equation which is

$$\left(\frac{m}{x} + \frac{n}{\alpha-x}\right)^2 - \frac{m}{x^2} + \frac{n}{(\alpha-x)^2} = 0. \tag{9}$$

Note that (8) also vanishes at $x = 0$ when $f(x) = 0$ which does not correspond to the minimum value that we are trying to find since $\mu(0) = \infty$. Multiplying both sides of (9) by $x^2(\alpha-x)^2$, simplifying and rearranging terms, we get the quadratic equation

$$(n-m)x^2 + 2m\alpha x + \alpha^2 (m^2 - m)/(n-m+1) = 0. \tag{10}$$

The roots of the equation are given by

$$\hat{x} = \frac{-m \pm \sqrt{mn/(n-m+1)}}{n-m}\,\alpha. \tag{11}$$

Observe that mn/(n — m + 1) > m^2 as long as 0 < *m* < 1. Therefore, for *n* > *m*, the acceptable positive root is given by

$$\hat{x} = \frac{-m + \sqrt{mn/(n - m + 1)}}{n - m}\,\alpha \tag{12}$$

and conversely for *n*<*m*. However, it can be seen from (10) that for n > m

$$2mx\hat{x} < a_2 (m - m_5)/(n - m + 1)$$

so that

$$\hat{x} < a/2. \tag{13}$$

Since $\hat{x}$ is much closer to zero than to a (Mitra, 1977), it is therefore imperative that

$$n > m \tag{14}$$

on empirical grounds. Combining (6) and (14) with (4), we may therefore conclude that an *f(x)* given by (4) with

$$A > 0, \quad n > m \quad \text{and } 0 < m < 1 \tag{15}$$

may be used to generate *l(x)* from (2).

Estimates of Parameters and Goodness of Fit

For the estimation of the parameters we begin by noting from (2) and (4) that

$$-ln\ l(x) = \frac{Ax^m}{(\alpha - x)^n}\,. \tag{16}$$

Next, we observe that taking logarithm of both sides of (15) produces a linear relationship as

$$in[- In\ l(x)] = InA + mlnx - n/n(a - x). \tag{17}$$

Equation (17) can then be used to generate the parametric estimates by the method of least squares if a is known or can be independently estimated. We have simplified this problem by assigning two values to a, namely, 95 and 100, since, for all practical purposes, the proportion of survivors of a birth cohort at any of those two ages can be regarded as insignificant or virtually equal to zero.

Next, in order to estimate the parameters of the multiple regression models, we have used the *l(x)* values for all the standard ages up to age 80 given in the Coale and Demeny's (1966) regional model life tables except the one at age 0. The reason for this exclusion is obvious since neither *ln(x)* nor *In*[—*M(x)]* can be computed at that point. It may be recalled that the model equation and the parameters have already been chosen in a manner subject to the condition that l(0) = 1. Accordingly, its exclusion from the data set for fitting the linear model does not seem to be of any consequence.

TABLE 1-THE PARAMETERS OF THE REGRESSION OF *In (-lnl(x)}* ON *x* AND a-x FOR SELECTED LEVELS OF NORTH FEMALE LIFE TABLES

Model Number	*e(0)*	*R2* a = 95	a = 100	*Estimates of Parameters* *A*	*m*	*n*	$\hat{X}$
1	20	.9869	.9838	14.60	.212	.769	19.0
5	30	.9893	.9866	10.77	.214	.796	18.6
9	40	.9909	.9883	9.32	.208	.854	17.7
13	50	.9921	9894	9.65	.193	.957	16.2
17	60	.9942	.9916	16.93	.143	1.190	14.4
21	70	.9977	.9959	53.52	.065	1.611	8.5

The life tables selected for this analysis covered virtually the entire range of the expectation of life while the selection of the region and the sex were arbitrarily made. It is gratifying to note that the square of the multiple correlation coefficient which, in this example, can be regarded as a measure of the goodness of fit, is quite large and ranges from .984 to .996 for a = 100 and from .987 to .998 for = 95. In general, the correlations are higher for a = 95 and they increase with the expectation of life.

Values of the model parameters shown in Table 1 reveal that the required conditions specified by (6) and (14) in section 2, namely, $0 < m < 1$ and $n > m$ have been met by these life tables. The reader may note that the parameter *n* increases with the life expectancy while the opposite is true for m with a minor exception for the level 1 life table. This is consistent with the fact that all other things remaining the same, a large value of *n* or a small value of *m* will produce a small value of (4) or a small negative exponent in (2). This in turn will generate a large value of *l(x)* thus reflecting a low level of mortality. For the other parameter A, it is possible to postulate a similar tendency although its empirical values generate a [U-shaped curve implying perhaps that all other things do not remain constant.

We would like to add that even though the model has been found to be highly satisfactory, the parameters for the last two levels, and especially for the last level, seem to have values that are quite different from what may be expected from the trends of the same for the previous levels. We would like to note in this context that the model life tables themselves that have been used for this study, have also been derived on the basis of certain statistical procedures. Therefore, these life tables may not be regarded as flawless and accordingly, the rapid changes in the parameters of our model may be attributed to the procedures used to construct them.

The age of minimum force of mortality shown in the last column also shows a sudden decline at level 21, consistent with changes in the other parameters. However, the declining trend of $\hat{x}$ is in conformity with the pattern observed earlier (Mitra, 1977), although the range is somewhat narrower in the regional life tables.

Summary and Conclusion

Previous attempts to graduate life table functions by mathematical formulas had been mildly to moderately successful. In this paper we have attempted to add to that list another model which has produced encouraging results. We have proposed a three parameter model of the survivorship function in which the life span has been assumed alternately as 95 and 100 years. The reproducibility of the model measured by the square of the multiple correlation has exceeded .98 in all the North female life tables selected from Coale and Demeny's regional model life tables. The parameters, as could be expected, are quite sensitive to the mortality levels and the trends of their variations have predictable patterns. It is apparent that the values of the parameters can be suitably changed to generate series of life tables.

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